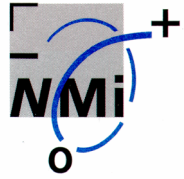


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# Use of error modelling in evaluating measurement uncertainty

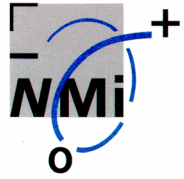
Adriaan M.H. van der Veen  
Nederlands Meetinstituut  
Maurice G. Cox  
National Physical Laboratory



## “Mainstream GUM” (1)

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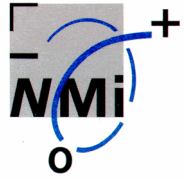
1. Express measurand  $Y$  mathematically in input quantities  $X_i$ ; function  $f$  should contain all variables that could contribute significantly to the measurement uncertainty
2. Determine all  $x_i$
3. Determine all  $u(x_i)$
4. Evaluate covariances



## “Mainstream GUM” (2)

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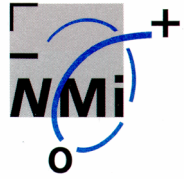
5. Calculate  $Y$
6. Calculate the combined standard uncertainty
7. Select coverage factor  $k$
8. Report expanded uncertainty  $U$



## Step 1 – Modelling

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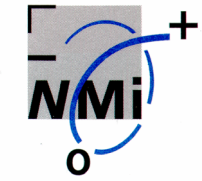
- Purpose: to translate measurement process into mathematical formalisms
- Formalisms required for processing numerical data
- Model should encompass all influencing factors
- Model relates influencing factors to measurand



## Modelling process

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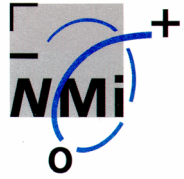
- Different approaches possible
- Often an augmentation of an existing relationship defining the measurand works fine
- 'Cascading' approach (e.g. Eurachem Guide) keeps maths simple, unless ... correlations come into play



## Error modelling

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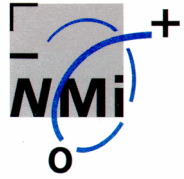
- Systematic approach to the problem
- In particular useful when starting from scratch
- Technique is very
  - flexible
  - robust
  - forces to state assumptions



## Error and uncertainty (I)

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- Existence of *errors* reason for having *uncertainty*
- Statistical treatment of errors provides corrections and standard uncertainties
- A single value for an error does not tell much about the uncertainty component ...
- ... but the behaviour does



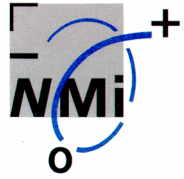
## Definition

---

- An error is defined as the departure of a variable from its expectation
- Expectation is defined as the result of

$$\mu = \int_{-\infty}^{\infty} xf(x)dx$$

- Relation with a true value requires an additional assumption!



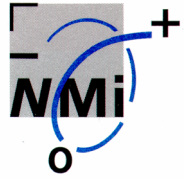
## Error and uncertainty (II)

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- Relationship is given through the variance

$$\text{Var}(X) = E\left((X - \mu)^2\right) = E(\varepsilon^2)$$

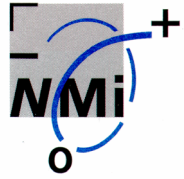
- Pdf describes behaviour of errors



## Example

- Given
  - $\langle q \rangle = 10$
  - $\text{Var}(q) = \sigma^2 = 2$
- Estimates
  - $m$  10.153
  - $s^2$  1.49

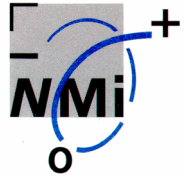
Drawing ( $i$ )	$q_i$	$\delta q_i$
1	8.22	-1.78
2	10.63	0.63
3	10.82	0.82
4	8.71	-1.29
5	10.51	0.51
6	11.27	1.27
7	11.66	1.66
8	9.19	-0.81
9	11.33	1.33
10	9.19	-0.81



## What does the GUM tell us?

---

- Propagation of uncertainties
  - clause 5
  - useful when directly modelling uncertainties
- Propagation of errors
  - clause E.3
  - basis of the GUM
  - conversion of an error model into an uncertainty model



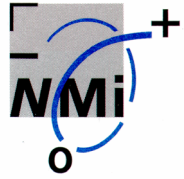
## Error modelling

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- Each quantity  $q$  can be expressed in its expectation and its error, e.g.

$$q = \langle q \rangle + \delta q,$$

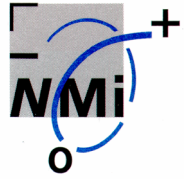
- Using the error propagation formula, the uncertainty in  $q$  equals the uncertainty of its error



## Case: degrees of equivalence

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- Given a laboratory comparison operated with a batch of artefacts
- A degree of equivalence is defined as
  - the difference (of a lab's result) from the reference value
  - the uncertainty thereof at 95% level of confidence



## Expression for the deviation

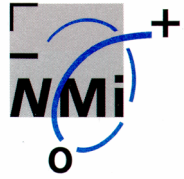
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- For a laboratory  $i$

$$D_i = x_i - x_{ref},$$

- where  $x_i$  denotes the laboratory result and  $x_{ref}$ .
- Between 2 laboratories  $i$  and  $j$

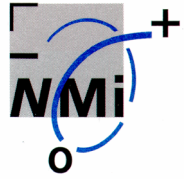
$$D_{ij} = (x_i - x_{ref}) - (x_j - x_{ref}) = x_i - x_j,$$



# Modelling

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- It seems that both definitions are fine for expressing the uncertainty of  $D_i$  and  $D_{ij}$  respectively, but what about
  - batch inhomogeneity
  - stability of the artefacts



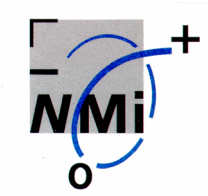
## Batch inhomogeneity

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- Affects  $D$  as a laboratory receives only one artefact
- Can be defined as a correction (to the reference value)

$$x_{bb,i} = \langle x_{bb,i} \rangle + \delta x_{bb,i}.$$

- each artefact has its own error (they are not necessarily equal!)

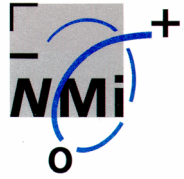


# Stability

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- Instability may be due to
  - intrinsic instability
  - storage conditions
  - (less than ideal) transport conditions
- Issues affect each artefact individually

$$x_{stab,i} = \langle x_{stab,i} \rangle + \delta x_{stab,i}.$$



## Consensus value

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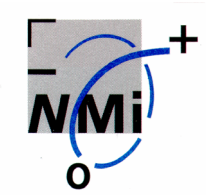
- A consensus value is used as reference

$$x_{cv} = \langle x_{cv} \rangle + \delta x_{cv}.$$

- Including all corrections leads to

$$x_{cv,i} = x_{cv} + x_{bb,i} + x_{stab,i} = \dots$$

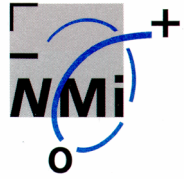
$$\dots = \langle x_{cv} \rangle + \delta x_{cv} + \langle x_{bb,i} \rangle + \delta x_{bb,i} + \langle x_{stab,i} \rangle + \delta x_{stab,i}.$$



## Assumptions

---

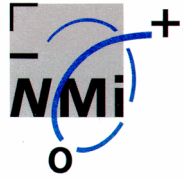
- So far, no assumptions have been made other than that inhomogeneity and instability of artefacts play a role
- Before further developing the model, it is now useful to make some assumptions/assertions



## Corrections

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- It is assumed that the homogeneity study is implemented in such a way, that the correction is 0.
- It is assumed that the stability study is implemented in such a way, that the correction is 0.



## Consensus value

---

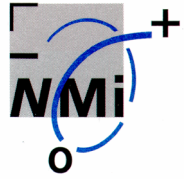
- Consensus value depends on laboratory results

$$x_{cv} = \sum w_i x_i.$$

- In terms of errors,

$$x_{cv} = \sum w_i \langle x_i \rangle + \sum w_i \delta x_i.$$

- where  $\delta x_i$  is the aggregated laboratory error term



## Deviation from $x_{cv}$

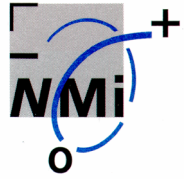
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- Model can now be expressed as

$$D_i = \langle x_i \rangle + \delta x_i - (\langle x_{cv} \rangle + \delta x_{cv} + \delta x_{bb,i} + \delta x_{stab,i}).$$

- where it should be noted that  $\delta x_{cv}$  and  $\delta x_i$  are (cor)related!
- Correlation can be dealt with by
  - substitution of expression for  $\delta x_{cv}$
  - evaluation of the covariance directly

$$u(x_i, x_{cv}) = w_i u^2(x_i),$$

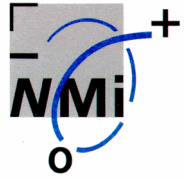


## Degree of equivalence $D_i$

- Expression for uncertainty

$$u^2(D_i) = u^2(x_i) + u^2(x_{cv}) + u_{bb}^2 + u_{stab}^2 - 2u(x_i, x_{cv}),$$

- follows directly from using the error propagation formula (GUM, E.3)



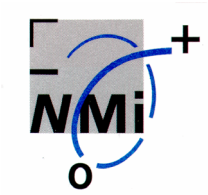
## Bilateral case

- Using the same methodology,

$$\begin{aligned} D_{ij} &= \left\{ x_i - \left( \langle x_{cv} \rangle + \delta x_{cv} + \delta x_{bb,i} + \delta x_{stab,i} \right) \right\} - \\ &\left\{ x_j - \left( \langle x_{cv} \rangle + \delta x_{cv} + \delta x_{bb,j} + \delta x_{stab,j} \right) \right\} = \dots \\ &\dots = x_i - \left( \delta x_{bb,i} + \delta x_{stab,i} \right) - x_j + \left( \delta x_{bb,j} + \delta x_{stab,j} \right). \end{aligned}$$

- which leads to

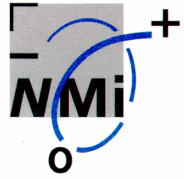
$$u^2(D_{ij}) = u^2(x_i) + u^2(x_j) + 2u_{bb}^2 + 2u_{stab}^2.$$



## Important notes on $D_{ij}$

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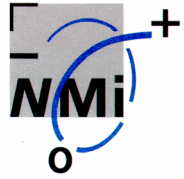
- Expectations of consensus value vanish
- Error terms of consensus value cancel
- Between-bottle variation enters *twice*
- Instability also enters *twice*



## Elaboration on $x_{CV}$

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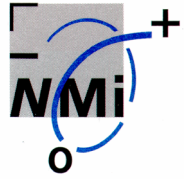
- Error term is aggregated (components:  $\delta x_i$ )
- Error terms  $\delta x_i$  also aggregated
- Aggregation irrelevant so far
- However: when combining results from different comparisons, aggregated terms become crucial



# Aggregation and de-aggregation

---

- Aggregation
  - Useful in many practical applications
  - GUM allows aggregated terms in the model
- De-aggregation
  - Necessary to address covariance-issues
  - Process answers the question: “What is contained in it?”
  - Knowledge counts, not the magnitude of the uncertainties!

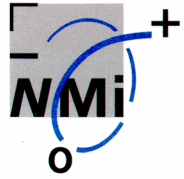


## De-aggregating $\delta x_{CV}$

- Separation in a within- and between laboratory term

$$\delta x_i = \delta x_{(L)i} + \frac{1}{n_i} \sum \delta x_{(r)i},$$

- Between-laboratory term still aggregated
- Uncertainty components become inseparable when  $n_i = 1$



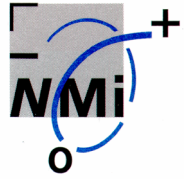
## Expression for $\delta x_{cv}$

- De-aggregation leads to

$$\delta x_{cv} = \sum w_i \delta x_{(L)i} + \sum w_i \left( \frac{\sum \delta x_{(r)i}}{n_i} \right).$$

- and finally to

$$u^2(x_{cv}) = \sum w_i^2 u^2(x_{(L)i}) + \sum w_i^2 \left( \frac{\sum u^2(x_{(r)i})}{n_i^2} \right),$$

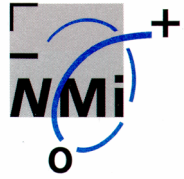


## Relationship with ISO 5725

- Assuming  $n_i = n$ ,
- each  $u(\delta x_{(L)}) = s_L$ ,
- each  $u(\delta x_{(L)}) = s_r$ ,

$$u^2(x_{cv}) = s_L^2 \sum w_i^2 + \left( \frac{s_r^2}{n} \right) \sum w_i^2.$$

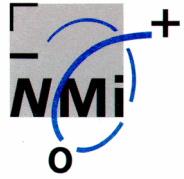
- and if all weights are equal ( $= 1/p$ ), the one-way ANOVA expression from ISO 5725-2 is obtained



## “Top-down” vs. “bottom-up”

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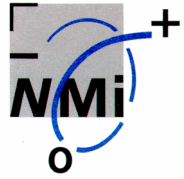
- Distinction is not very useful at technical level
- Approaches should be consistent
- Aggregated terms have great benefits, but are difficult to handle if it is unknown what is contained
- De-aggregation is key to demonstrating compliance with ISO/IEC 17025



## Concluding remarks (1)

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- Error modelling is a useful technique for constructing uncertainty models
- Main benefits are
  - ease of use
  - clear understanding of the assumptions made
  - often shortest route to the desired uncertainty model



## Concluding remarks (2)

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- Error modelling in particular useful when
  - starting from scratch
  - augmenting existing models
- Mechanism of conversion is by means of error propagation (GUM, E.3), rather than uncertainty propagation